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TWO GLOBAL CHARACTERISTICS OF THE GRAPHS

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Abstract
Given adjacent vertices v and w of a graph G = (V ,E) , the Randic weight of the edge {v, w}is the number $R(\{v,w\})$ *. The Randic weight of a graph G, R(G) is the sum of the weights of its edges. This weight was first introduced by M. Randic [5].*
B. Bollobas and P. Erdös [2]defined, for $\alpha \in R, \alpha \neq 0$ *, the weight* $w_{\alpha}(e)$ *of an edge e={v, w} of a graph to be* $w_{\alpha}(e) = (d(v)d(w))^{\alpha}$ *. Thus* $w_{-\frac{1}{2}}(e)$ *is the Randic weight of the edge. For the graph G they defined* $w(G) = \sum_{e \in E(G)} w(e)$ *and* $w_{\alpha}(G) = \sum_{e \in E(G)} w_{\alpha}(e)$ *.*
The special degree of the node $v \in V$ *, introduced by M.Cocan and V. Proşcanu [3], is a number GS(v) =* $(s_1 s_2 ... s_n)_{(b)}$ *calculated in b basis,* $b \in N^*, n > 1$ *.*
The special degree is a global feature of the node, which depends on the entire graph; it is a number that expresses how ''strong'' the respective node is, depending on its degree and the degrees of all its descendants.
M.Cocan [4] introduced a global characteristic of a graph, named the graph connection power. He will determine this value by using the concept of special degree of a graph vertex.
Both the Randic weight and the connection power are global characteristics of a graph. The present paper aims at constructing a comparative study of the Randic weight of a connected graph and the graph connection power.

Keywords*: connected graph, special degree, incidence matrix, Randic weight graph*

1. INTRODUCTION

The present paper aims at constructing a comparative study of the Randic weight of a connected graph and the graph connection power. Both the Randic weight and the connection power (or special degrees vector) are global characteristics of a graph.

We define a graph *G* as an orderer paar *G* = (*V* ,*E*), where *V* and *G* are sets, *V* is the vertex set of *G* and *E* is the edge set of *G*. The elements of *V* are vertices and the elements of *G* are edges. *G* is *null* if $V = \emptyset$ and empty if $E = \emptyset$.

The degree, $d_G(v)$, in *G* of a vertex $v \in V$ is the sum of the numbers the links incident on *v* and double numbers loops incident on *v*. We may delete the subscript in this notation if no ambiguity emerges. Vertices of degrees 0 are isolated. Two distinct vertices are adiacent if they are incident on a common edgs. Adjacent edges and adjacent vertices are sometimes described as neighbours.

In [1] is defined the cocycle $\partial_G v$ of a vertex in a graph *G* as the set of all links incident on *v*. The cocycle $\partial_G S$ of a set *S* of vertices is the sum of the cocycles of those vertices. Thus $\partial_G\{v\}$ = $\partial_G v$; $\partial_G v$ is a vertex cocycle. The symbol ∂_G may be replaced by ∂ if no ambiguity results.

2. THE RANDIC WEIGHT OF THE EDGE IN A GRAPH

Given adjacent vertices *v* and *w* of a graph *G* = (*V* ,*E*) , the Randic weight or simply weight of the edge {*v*, *w*}is $R(\{v,w\}) = (d(v)d(w))^{-\frac{1}{2}}$, where *d*(*v*) and *d*(*w*) are the degrees of *v* and *w*. The Randic weight or simply weight of a graph *G*, *R*(*G*) is the sum of the weights of its edges. This weight was first introduced by M. Randic in 1975.

B. Bollobas and P. Erdös [1998] defined, for $\alpha \in R$, $\alpha \neq 0$, the weight $w_\alpha(e)$ of an edge *e*={*v*, *w*} of a graph to be $w_\alpha(e) = (d(v)d(w))^\alpha$. Thus $w_1(e)$ is simply the weight $w(e)$, and $w_{-\frac{1}{2}}(e)$ is the Randic weight of the edge.

For the graph *G* they defined $w(G) = \sum_{e \in E(G)} w(e)$ and $w_\alpha(G) = \sum_{e \in E(G)} w_\alpha(e)$.

In [2] is present a interesting set of the results of the weight $w_\alpha(e)$ and implicit of the Randic weight.

3. SUCCESSOR (DESCENDANT) OF k^{th} ORDER , $k \in N$, OF A NODE IN A GRAPH

Definition 1 Node $w \in V(G)$ is called *descendant* of node *v*, if *w* is accessible from *v*, by a links, in *G*. If the length of the link is 1 (it consists of only one edge), then we say that *w* is a *direct (immediate) descendant* of *v*.

Let us note:

$$Succ_d(v) := \{w \mid w \in V, \{v,w\} \in M\};$$

Case 1 Graph *G* does not contain cycles.

We define:

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$$Succ^{(0)}(v) := \{v\};$$
$$Succ^{(1)}(v) := Succ_d(v);$$
$$Succ^{(k)}(v) := Succ_d(Succ^{(k-1)}(v)) \setminus Succ^{(k-2)}(v), k \geq 2.$$

where:

$$Succ_d(V_1) = \bigcup_{v \in V_1} Succ_d(v), \forall V_1 \subset V.$$

Then we have:

$$\left|Succ_d(v)\right| = \left|\partial_G v\right|;$$
$$\left|Succ^{(2)}(v)\right| = \left|\partial_{G_{V(G)-\{v\}}}(Succ_d(v))\right|;$$
$$\left|Succ^{(k)}(v)\right| = \left|\partial_{G_{V(G)-\bigcup_{j=0}^{k-2} Succ^{(j)}(v)}}(Succ^{(k-1)}(v))\right|, k \geq 2.$$

where $|X|$ represents the number of elements of the X set, and G_{V_1} represents the subgraph of graph G induced by the set of nodes $V_1, V_1 \subset V$.

Definition 2 The elements of the set $Succ^{(k)}(v)$ are k^{th} order descendants of node v. For $k = 1$, we obtain the direct descendants.

Example. $G = (V, E)$, $V = \{v_1, v_2, ..., v_{13}\}$.

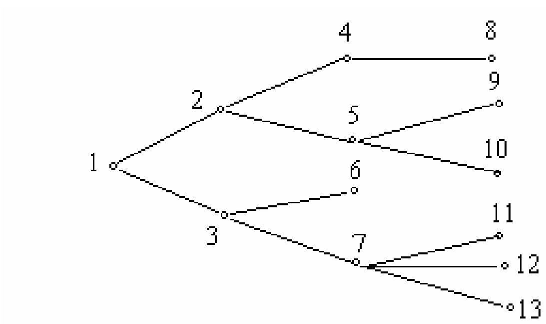

Figure 1

$Succ_d(v_1) = \{v_2, v_3\}$; $\partial_G v_1 = \{\{v_1, v_2\}, \{v_1, v_3\}\}$;

$Succ^{(2)}(v_1) = Succ_d(Succ_d(v_1)) \setminus Succ^{(0)}(v_1) =$

$Succ_d(\{v_2, v_3\}) \setminus \{v_1\} = \{v_1, v_4, v_5, v_6, v_7\} \setminus \{v_1\} = \{v_4, v_5, v_6, v_7\}$;

$\partial_{G_{V(G)\setminus\{v_1\}}}(Succ_d(v_1)) = \partial_{G_{\{v_2, v_3, ... v_{13}\}}}(\{v_2, v_3\}) = \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_6\}, \{v_3, v_7\}\}$;

$$Succ^{(3)}(v_1) = Succ_d(Succ^{(2)}(v_1) \setminus Succ^{(1)}(v_1) =$$
$$= Succ_d(\{v_4, v_5, v_6, v_7\}) \setminus \{v_2, v_3\} = \{v_2, v_3, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\} \setminus \{v_2, v_3\} =$$
$$= \{v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\};$$
$$\partial_{G_{V(G)\setminus(\{v_1\}\cup Succ^{(1)}(v_1))}}(\{v_4, v_5, v_6, v_7\}) = \partial_{G_{\{v_4, v_5, \ldots, v_{13}\}}}(\{v_4, v_5, v_6, v_7\}) =$$
$$= \{\{v_4, v_8\}, \{v_5, v_9\}, \{v_5, v_{10}\}, \{v_7, v_{11}\}, \{v_7, v_{12}\}, \{v_7, v_{13}\}\};$$
$$Succ^{(k)}(v_1) = \emptyset;\ k \geq 4;$$
....
$$Succ_d(v_{13}) = \{v_7\}; \partial_G v_{13} = \{v_7, v_{13}\}; \left|\partial_G v_{13}\right| = 1;$$
$$Succ^{(2)}(v_{13}) = \{v_3, v_{11}, v_{12}\}; \left|\partial_{G_{V(G)\setminus\{v_{13}\}}} v_7\right| = 3;$$
$$Succ^{(3)}(v_{13}) = \{v_1, v_6\}; \left|\partial_{G_{V(G)\setminus\{v_7, v_{13}\}}}(\{v_3, v_{11}, v_{12}\})\right| = 2;$$
$$Succ^{(4)}(v_{13}) = \{v_2\}; \left|\partial_{G_{V(G)\setminus\{v_3, v_7, v_{11}, v_{12}, v_{13}\}}}(\{v_1, v_6\})\right| = 1;$$
$$Succ^{(5)}(v_{13}) = \{v_4, v_5\}; \left|\partial_{G_{V(G)\setminus \bigcup_{k=0}^{3} Succ^{k}(v_{13})}}(v_2)\right| = 2;$$
$$Succ^{(6)}(v_{13}) = \{v_8, v_9, v_{10}\}; \left|\partial_{G_{V(G)\setminus \bigcup_{k=0}^{4} Succ^{(k)}(v_{13})}}(\{v_4, v_5\})\right| = 3;$$
$$Succ^{(k)}(v_{13}) = \emptyset, \quad k > 6.$$

Case 2 The graph has cycles.

If G has cycles, then there are two possibilities:

a) $(\exists)\ w_1, w_2 \in Succ^{(k)}(v),\ k \in N^*$ so that $\{w_1, w_2\} \in E$. This possibility is represented in figure 4.

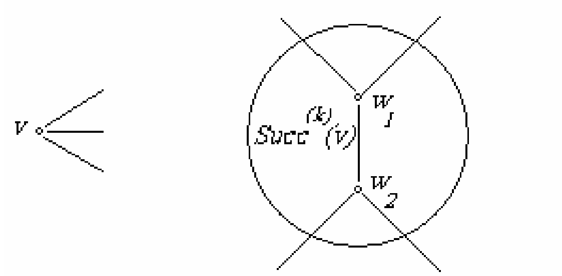

Figure 2

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$Succ^{(0)}(v) := \{v\};$
$Succ^{(1)}(v) := Succ_{(d)}(v) = \{w | w \in V, \{v, w\} \in E\};$
And
$$|Succ^{(k+1)}(v)| = |Succ_{(d)}(Succ^{(k)}(v) \setminus Succ^{(k-1)}(v)| + \\ + |\{(w_i, w_j) / w_i, w_j \in Succ^{(k)}(v), w_i \neq w_j, \{w_i, w_j\} \in E|$$
or
$$|Succ^{(k+1)}(v)| = \left| \partial_{G_{V(G) \setminus \bigcup_{j=0}^{k-1} Succ^{(j)}(v)}} (Succ^{(k)}(v)) \right| + 2p,\ k \geq 2$$
where p is the number of distinctive nodes in $Succ^{(k)}(v)$ which are joined by edges in G.

4. THE SPECIAL DEGREE OF A NODE IN A GRAPH

Take the connected graph $G = (V, E)$, $/V/= n$, $/E/= m$, whose incidence matrix $\mathbf{A} \in \boldsymbol{M}_{m,n}\{(0,1)\}$ has the form:

$$A = \begin{array}{c} \\ e_{j_1} \\ e_{j_2} \\ . \\ . \\ . \\ e_{j_m} \end{array} \begin{array}{c} \begin{array}{cccc} v_{i_1} & v_{i_2} & . \ . \ . & v_{i_n} \end{array} \\ \begin{pmatrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ & & & & & \\ & & & & & \\ & & & & & \\ . & . & . & . & . & . \end{pmatrix} \end{array}$$

Definition 3 The special degree of the vertex $v \in V$, is a number $GS(v) = (s_1 s_2 ... s_n)_{(b)}$ calculated in b basis, $b \in N^*, n > 1$, where:

s_1 - represents for the degree of the vertex v;

s_2 - represents for the degrees sum of the direct descendants of the vertex v (the neighbouring vertices of the vertex v), after the vertex v has been eliminated;

s_3 - represents for the degrees sum of the direct descendants of the direct descendants (in other words the degrees sum of the secondary descendants) of the vertex v, after the direct descendants of v have also been eliminated, and so on.

We suppose that $n \geq m$.

This concept was introduced by M.Cocan and V. Proşcanu [3].

The special degree is a global feature of the vertex, which depends on the entire graph; it is a number that expresses how "strong" the respective vertex is, depending on its degree and the degrees of all its descendants.

The special degree can also be extended on multigraphs and unconnected graphs.

M.Cocan [1999] introduced a global characteristic of a graph, named the graph connection power. He will determine this value by using the concept of special degree of a graph vertex.

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5. A NECESSARY AND SUFFICIENT CONDITION FOR THE ISOMORPHISM OF GRAPHS USING THE SPECIAL DEGREES

Theorem 2 The necessary and sufficient condition for two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be isomorphic is that $GS_1 = GS_2$, where $GS_i (i = 1,2)$ represents the sets of the special degrees of the graph nodes G_i $(i = 1,2)$ with the components in increasing or decreasing order.

Proof. Sufficiency. From $GS_1 = GS_2$ it follows that $n_1 = n_2$, where $n_1 = |V_1|$, $n_2 = |V_2|$.

Take $V_1 = \{v_1^1, v_2^1, ..., v_n^1\}$ and $V_2 = \{v_1^2, v_2^2, ..., v_n^2\}$, $n = n_1 = n_2$.

From $GS_1(v_i^1) = GS_2(v_j^2)$ taking into account the fact that:

$$GS_1(v_i^1) = s_{1,i}^1 b^{n-1} + s_{2,i}^1 b^{n-2} + ... + s_{n,i}^1$$

$$GS_2(v_j^2) = s_{1,j}^2 b^{n-1} + s_{2,j}^{21} b^{n-2} + ... + s_{n,i}^2$$

it follows that

$$s_{k,i}^1 = s_{k,j}^2, \quad k = \overline{1,n},$$

which enables us to define the isomorphism $\varphi : V_1 \to V_2$ by $\varphi \ (v_i^1) = v_j^2$.

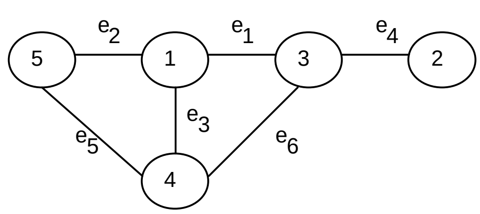
The equality $s_{1,i}^1 = s_{1,j}^2$ expresses the fact that the degree of the node v_i^1 is equal to degree of node v_i^2.

Necessity. Suppose that the graphs G_1 and G_2 are isomorphic; then $\mathbf{M_1} = \mathbf{M_2}$, $\mathbf{M_i}$ $(i = 1,2)$ being the maximal incidence matrices of the graphs G_i $(i = 1,2)$, and next $GS_1 = GS_2$.

6. EXAMPLES

The programme for determining the special degrees of the nodes and the maximal incidence matrix, devised in Delphi, provided the following results for the graphs G_i (i =1,...,5).

1) G_1: $n = 5$, $m = 6$;



$GS(v_1) = GS(v_4) > GS(v_3) > GS(v_5) > GS(v_2)$.

Figure 3

2) G_2: $n = 6$, $m = 9$;

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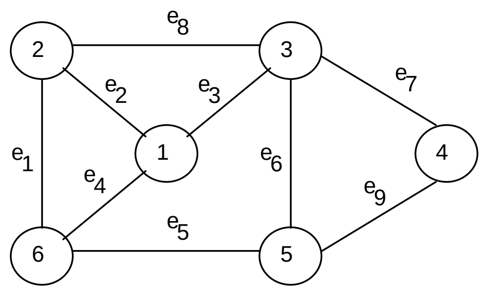
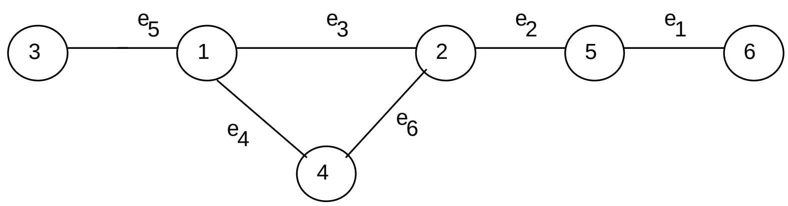


$GS(v_3)>GS(v_2)=GS(v_1)>GS(v_6)>GS(v_5)>GS(v_4)$.

Figure 4

3) G_3: $n = 6$, $m = 6$;

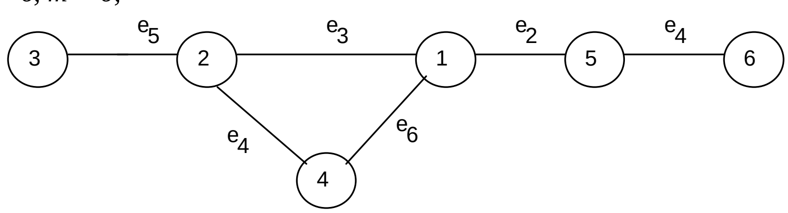


$GS(v_2)>GS(v_1)>GS(v_4)>GS(v_5)>GS(v_3)>GS(v_6)$

Figure 5

Node v_2 is “stronger” than node v_1 due to node v_6.

4) G_4: $n = 6$, $m = 6$;

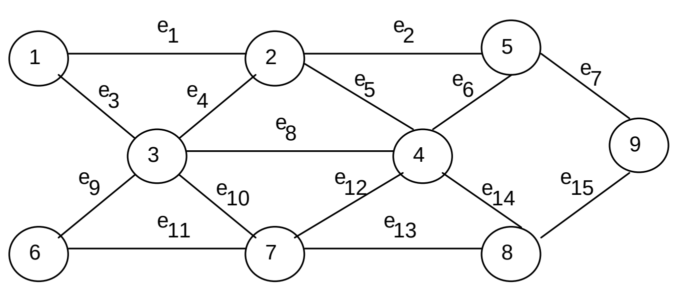


$GS(v_1)>GS(v_2)>GS(v_4)>GS(v_5)>GS(v_3)>GS(v_6)$

Figure 6

In comparison to example 3, the nodes v_1 and v_2 were interchanged and the algorithm „noticed” this.

5) G_5: $n = 9$, $m = 15$;



$GS(4)>GS(3)>GS(2)=GS(7)>GS(5)=GS(8)>GS(1)=GS(6)>GS(9)$.

Figure 7

Node v_4 is “stronger” than node v_3 due to node v_9.

7. A COMPARATIVE STUDY OF THE RANDIC WEIGHT OF A CONNECTED GRAPH AND THE GRAPH CONNECTION POWER

We mention a few representative aspects of these two characteristics of the graphs.

(*i*) The special degree and Randic weight are two global characteristics of a graph, because they contain the entire information of the graph.

(*ii*) The Randic weight of a graph *G*, of the order *n*, having no isolated nodes satisfies inequality $R(G) \geq n-1$; the inequality is equality for the star graphs.

(*iii*) The weight of the graph *G*, with the number the edges *m*, satisfy inequality $w_\alpha(G) \geq m(\frac{\sqrt{8m+1}-1}{2})^{2\alpha}$, if $\alpha \in [0,1)$; for the Randic weight $\alpha = -\frac{1}{2}$, therefore $R(G) \geq \frac{\sqrt{8m+1}+1}{4}$; the inequality is equality if and only if $m = \binom{n}{2}$, therefore if $G = K_n$ is a complete graph of the order *n* and possibly isolated nodes.

(*iv*) The *i*-component (digit) of the special degree of the node of the unoriented graph satisfy following property $s_i \leq max\ (m, n)\ i = 1, 2, \ldots, n.$

(*v*) The special degree of a node is calculated by a recursive in-depth following of the graph.

(*vi*) For calculating the special degree *GS(v)* in basis *10* the following formula is used:

$$GS(v) = s_1 \cdot b^{n-1} + s_2 \cdot b^{n-2} + \ldots + s_n \cdot b^0 .$$

(*vii*) The special degree is a global feature of the node, which depends on the entire graph; it is a number that expresses how „strong" the respective node is, depending on its degree and the degrees of all its descendants.

(*viii*) The special degree can also be extended on multigraphs graphs.

(*ix*) The necessary and sufficient condition for a graph $G = (V, E)$ to be connected is that $\sum_{i=1}^{n} GS_i(v) = m,\ \forall v \in V$, where $GS(v) = (s_1, s_2, \ldots, s_n)$, $GS_i(v) = s_i, (i = \overline{1,n})$.

(*x*) The necessary and sufficient condition for a graph $G = (V, E)$, $|V| = n$ to be a elementary chain $\delta = \{v_1, v_2, \ldots, v_n\}$ is that:

$$GS(v_i) = GS(v_{n-i+1}) = ((i-1)*2, (n-2*i+1, (i)*0)_{(b)},\ i = \overline{1,k}, k = \left[\frac{n}{2}\right].$$

(*xi*) The number of connected components in the graph *G* = (*V*, *E*), is greater or equal than the number of distinct values from $\{\sum_{i=1}^{n} GS_i(v) \mid v \in V\}$.

(*xii*) An algorithm for establishing the connected components in a graph is the following:

Step 1 Determine $GS(v) = (s_1(v), s_2(v), \ldots, s_n(v))$, $\forall v \in V$;

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Step 2 Calculate the values $N(v) = \sum_{i=1}^{n} s_i(v), \forall v \in V$;

Step 3 Determine the distinct values $N_1, N_2, ..., N_s$ of *the vector's components* $(N(v_1), N(v_2), ..., N(v_n))$ of the graph *G*.

Step 4 Determine the partition $V_i\,(i = \overline{1,s})$ *of the vertices set of the graph*:

$$V = \bigcup_{i=1}^{n} V_i,\ V_i \cap V_j = \varnothing, (i \neq j), |V_i| = N_i, i = \overline{1,s};$$

$$V_i = \{v \in V \mid N(v) = N_i\}, i = \overline{1,s}\,;$$

Step 5 Calculate $c_i = \dfrac{card(V_i)}{N_i + 1}$ for every $i = \overline{1,s}$;

Step 6 Write the number $c = \sum_{i=1}^{s} c_i$ *represents the number of connected components of the graph G.*

(*xiii*) Application: two isomorphic graphs (G_1 and G_2).

1. $G_1 = (V_1, E_1)$,
$V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$,
$E_1 = \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_3, v_7\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_7\}\}$;

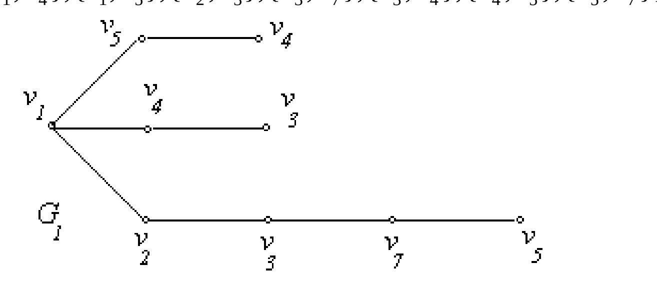

Figure 8

The special degrees of the nodes for the graph G_1 are:

$GS(v_4) = 366660000$; $GS(v_1) = 357830000$; $GS(v_5) = 357830000$;

$GS(v_3) = 348820000$; $GS(v_2) = 247970000$; $GS(v_7) = 247970000$;

$GS(v_6) = 0$; $GS(v_8) = 0$; $GS(v_9) = 0$.

M\V	V4	V1	V5	V3	V2	V7	V6	V8	V9
m2	1	1	0	0	0	0	0	0	0
m8	1	0	1	0	0	0	0	0	0
m7	1	0	0	1	0	0	0	0	0
m3	0	1	1	0	0	0	0	0	0
m1	0	1	0	0	1	0	0	0	0
m6	0	0	1	0	0	1	0	0	0
m4	0	0	0	1	1	0	0	0	0
m5	0	0	0	1	0	1	0	0	0

Figure 9 The maximal incidence matrix of the graph G_1

2. $G_2 = (V_2, E_2)$,

$$V_2 = \{v_{1'}, v_{1'}, v_{2'}, v_{4'}, v_{4'}, v_{5'}, v_{6'}, v_{7'}, v_{7'}\},$$

$$E_2 = \{\{v_{1'}, v_{2'}\}, \{v_{1'}, v_{6'}\}, \{v_{1'}, v_{7'}\}, \{v_{2'}, v_{4'}\}, \{v_{4'}, v_{5'}\}, \{v_{4'}, v_{7'}\}, \{v_{5'}, v_{6'}\}, \{v_{6'}, v_{7'}\}\};$$

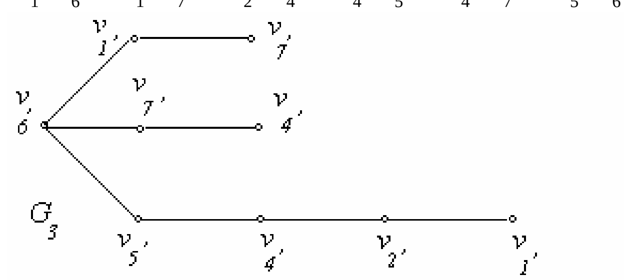

Figure 10

The special degrees of the nodes for the graph G_2 are:

$$GS(v_{7'}) = 366660000\,; \quad GS(v_{6'}) = 357830000\,; \quad GS(v_{1'}) = 357830000\,;$$

$$GS(v_{4'}) = 348820000\,; \quad GS(v_{2'}) = 247970000\,; \quad GS(v_{5'}) = 247970000\,;$$

$$GS(v_{3'}) = 0\,; \qquad GS(v_{8'}) = 0\,; \qquad GS(v_{9'}) = 0\,.$$

M\V	V7	V6	V1	V4	V5	V2	V3	V8	V9
m8	1	1	0	0	0	0	0	0	0
m2	1	0	1	0	0	0	0	0	0
m5	1	0	0	1	0	0	0	0	0
m6	0	1	1	0	0	0	0	0	0
m7	0	1	0	0	1	0	0	0	0
m1	0	0	1	0	0	1	0	0	0
m4	0	0	0	1	1	0	0	0	0
m3	0	0	0	1	0	1	0	0	0

Figure 11 The maximal incidence matrix for the graph G_2

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